CYCLICITY OF SOME SYMMETRIC NILPOTENT CENTERS

ISAAC A. GARCÍA[1]

Abstract. In this work we present techniques for bounding the cyclicity of a wide class of monodromic nilpotent singularities of symmetric polynomial planar vector fields. The starting point is identifying a broad family of nilpotent symmetric fields for which existence of a center is equivalent to existence of a local analytic first integral, which, unlike the degenerate case, is not true in general for nilpotent singularities. We are able to relate so-called "focus quantities"" to the "Poincaré-Liapunov quantities" arising from the Poincaré first return map. When we apply the method to concrete examples, we show in some cases that the upper bound is sharp. Our approach is based on computational algebra methods for determining a minimal basis (constructed by focus quantities instead of by Poincaré-Liapunov quantities because of the easier computability of the former) of the associated polynomial Bautin ideal in the parameter space of the family. The case in which the Bautin ideal is not radical is also treated.

1. Introduction and main results

Knowledge of the local qualitative phase portrait near an isolated singularity $p_0 \in \mathbb{R}^2$ of a real analytic planar vector field $\mathcal{X}$ is an almost completely solved problem, see for example [7]. Only for the so-called monodromic singularities that problem remains open. We recall that p_0 is *monodromic* when nearby orbits of $\mathcal{X}$ rotate about p_0. Also it is well known after an independent proof in [8] and [15] that monodromic singularities only can be *centers* (having a punctured neighborhood filled with periodic orbits) or *foci* (having a punctured neighborhood filled with spiraling orbits). The *center problem* is the problem of distinguishing between a center or a focus at a monodromic singular point.

In this work we focus on *nilpotent* singularities which appears when the linear part $D\mathcal{X}(p_0)$ of $\mathcal{X}$ at p_0 is nonzero and has two zero eigenvalues. More precisely, we will deal with polynomial families of planar vector field $\mathcal{X}_\lambda$, with parameters $\lambda \in \mathbb{R}^M$, and having a monodromic nilpotent singularity. Several general techniques have been designed to theoretically solve the nilpotent center problem such as: (i) the generalized polar blow-up explained in [3, 11]); (ii) the normal form theory that transforms $\mathcal{X}_\lambda$, up to some order, into an orbitally conjugate Liénard canonical form [4], and (iii) to embed $\mathcal{X}_\lambda$ as the limit as $\varepsilon \to 0$ of a greater family $\mathcal{X}_{(\lambda,\varepsilon)}$ of nondegenerate centers at p_0, see [14] for the initial idea but take into account [10] as final corrected version.

2010 *Mathematics Subject Classification.* 37G15, 37G10, 34C07.

Key words and phrases. Monodromic singularity, nilpotent center, cyclicity, limit cycle.

The author is partially supported by a MINECO grant number MTM2014-53703-P and by a CIRIT grant number 2014 SGR 1204.

In this paper we are interested in obtaining the *cyclicity* of the nilpotent singularity p_0 inside family $\mathcal{X}_\lambda$, that is, we want to compute the maximum number of *limit cycles* (isolated periodic orbits of $\mathcal{X}_\lambda$) that can be made to bifurcate from p_0 under small perturbation of the parameters λ. For solving this cyclicity problem it is natural first to compute the Poincaré-Lyapunov quantities that arise from the coefficients in the Taylor expansion of the analytic Poincaré first return map. For the nilpotent monodromic singularities this is not an easy task, see [3, 11].

Although all the previously mentioned methods give solutions of the nilpotent center problem, in practice there are only a few families $\mathcal{X}_\lambda$ whose centers are known due mainly to the complex and massive computations needed. A crucial difference between nilpotent centers and *nondegenerate* centers (those centers for which the eigenvalues of $D\mathcal{X}(p_0)$ have nonzero imaginary part) is that a nilpotent center does not necessarily have a local analytic first integral. This difference causes further complications when trying to generalize the computational algebra techniques described in [19] and specifically designed for studying the cyclicity of nondegenerate centers in polynomial families to the nilpotent case. To overcome this problem we will work with a subset of nilpotent families $\hat{\mathcal{X}}_\lambda$ having a Poincaré-Lyapunov like property that characterizes the nilpotent centers of the family via the existence of a local analytic (or merely formal) first integral. Thus the characterization of this kind of nilpotent centers leads to a sequence of polynomials in λ called *focus quantities* whose vanishing provide us with necessary center conditions. The outcome is that the set of vector fields $\hat{\mathcal{X}}_\lambda$ with a nilpotent center at p_0 corresponds to an affine variety called the *center variety* in the parameter space. The real advantage is that the focus quantities are much easier to work with than the Poincaré-Lyapunov quantities because they can be computed efficiently only using algebraic manipulations without the need of quadratures. A far as we know the first systematic method for obtaining an upper bound on the cyclicity inside a broad class of nilpotent families $\hat{\mathcal{X}}_\lambda$ was performed in [13] for the planar differential systems $\dot{x} = y + P_{2m+1}(x, y; \lambda)$, $\dot{y} = Q_{2m+1}(x, y; \lambda)$, where P_{2m+1} and Q_{2m+1} are homogeneous polynomials of degree $2m + 1$ in x and y, and λ parameterizes the coefficients of the polynomials P_{2m+1} and Q_{2m+1}. In this sense, this work is a natural continuation and generalization of [13].

Let us consider polynomial families of planar vector fields $\hat{\mathcal{X}}_\lambda$ having a monodromic nilpotent singularity. After an appropriate affine change of variables and a time rescaling the singularity is placed at the origin and the linear part of its associated differential system is written in Jordan canonical form. More precisely the system is written in the form

$$\dot{x} = y + P(x, y; \lambda), \quad \dot{y} = Q(x, y; \lambda), \tag{1}$$

where P and Q are real polynomials in the phase variables $(x, y) \in \mathbb{R}^2$ without constant or linear terms and λ parameterizes its coefficients. We will call $\mathcal{X}_\lambda = (y + P(x, y; \lambda))\partial_x + Q(x, y; \lambda)\partial_y$ the associated vector field to (1).

The monodromy problem for analytic nilpotent singularities was solved in [6] by Andreev. We state now that result.

Theorem 1 ([6]). *For an analytic system of the form* (1) *with* $\lambda = \lambda^\dagger$ *and an isolated singularity at the origin let* $y = F(x)$ *be the unique solution of* $y + P(x, y; \lambda^\dagger) =$

0 such that $F(0) = F'(0) = 0$ and let

$$f(x) = Q(x, F(x); \lambda^\dagger) \qquad \text{and} \qquad \varphi(x) = (\partial P/\partial x + \partial Q/\partial y)(x, F(x); \lambda^\dagger).$$

Let $a \neq 0$ and $\alpha \geq 2$ be such that $f(x) = ax^\alpha + \cdots$.
When φ is not identically zero let $b \neq 0$ and $\hat{\beta} \geq 1$ be such that $\varphi(x) = b\, x^{\hat{\beta}} + \cdots$.
Then the origin of (1) is monodromic if and only if $\alpha = 2n - 1$ is an odd integer, $a < 0$, and one of the following conditions holds:
(i) $\varphi(x) \equiv 0$
(ii) $\hat{\beta} \geq n$
(iii) $\hat{\beta} = n - 1$ *and* $b^2 + 4an < 0$.

Definition 2. The *Andreev number* n of the monodromic singular point at the origin of system (1) is the integer $n \geq 2$ given in Theorem 1.

It is easy to convince yourself that, when studying analytic monodromic nilpotent singularities, $(1, n)$-quasihomogeneous polynomials appear in a natural way.

Definition 3. A polynomial $R_k \in \mathbb{R}[x, y]$ is a (p, q)–quasihomogeneous polynomials of weighted degree k if $R_k(\mu^p x, \mu^q y) = \mu^k R_k(x, y)$ for all $\mu \in \mathbb{R}$. In consequence we get $R_k(x, y) = \sum_{pi+qj=k} a_{ij} x^i y^j$ for certain coefficients $a_{ij} \in \mathbb{R}$. On the other hand, a vector field $\mathcal{X}_i = P_{i+p}\partial_x + Q_{i+q}\partial_y$ is a (p, q)–quasihomogeneous polynomial vector field of weighted degree i if its components P_{i+p} and Q_{i+q} are (p, q)–quasihomogeneous polynomials of weighted degrees $i + p$ and $i + q$, respectively.

Our starting point is the work [1]. In that paper the authors study the nilpotent center problem for the following analytic family of planar vector fields

$$\mathcal{X} = \sum_{i=0}^{\infty} \mathcal{X}_{q-p+2is} \tag{2}$$

where $\mathcal{X}_k$ denotes a (p, q)-quasihomogeneous vector field of weighted degree k and the following three conditions hold for the full family:
(i) p and q are positive odd integers without common factors and with $p \leq q$;
(ii) $s = np - q \geq 1$ with $n \geq 2$ an integer;
(iii) $\mathcal{X}_{q-p} = y\partial_x$ and $\mathcal{X}_{q-p+2s} = \mathcal{X}_{(2n-1)p-q} = A(x, y)\partial_x + B(x, y)\partial_y$ with $B(1, 0) < 0$. Actually we can take $B(1, 0) = -1$ without loss of generality which means that the monomial $-x^{2n-1}$ is always present in $B(x, y)$.

Remark 4. Let us consider for a while family (2) but changing property (i) by the following one: p and q are positive integers without common factors, $p \leq q$, and either p or q even. Then the origin is a nilpotent center because it is monodromic and time-reversible since $\mathcal{X}(x, -y) = -\mathcal{X}(x, y)$.

Remark 5. Family (2) contains the families studied in [13], that is, the polynomial nilpotent systems (1) with nonlinearities P and Q given by homogeneous polynomials of degree $2n - 1$. That subfamily is given by (2) with $p = q = 1$ and $\mathcal{X}_{q-p+2is} \equiv 0$ for $i \geq 2$.

For the origin of family (2) we first prove the following monodromic structure.

Proposition 6. *Family (2) possesses a monodromic nilpotent singularity at the origin with associated Andreev number n and $\hat{\beta} \geq n$.*

Definition 7. We say that family (1) is *symmetric* (with respect to the origin) if it is invariant under the involution $(x,y) \mapsto (-x,-y)$. In other words, $P(-x,-y;\lambda) = -P(x,y;\lambda)$ and $Q(-x,-y;\lambda) = -Q(x,y;\lambda)$.

Proposition 8. *Family* (2) *is symmetric.*

Remark 9. Clearly family (2) does not contain all the symmetric vector fields having a monodromic nilpotent singularity at the origin. The reason is because conditions (i) and (ii) on that family impose severe restrictions on the possible values of p, q and n except in the case $p = q = 1$. Just in this case family (2) reduces to the analytic family given by (1) where the components are $P(x,y;\lambda) = \sum_{i\geq 1} P_{2i(n-1)+1}(x,y;\lambda)$ and $Q(x,y;\lambda) = \sum_{i\geq 1} Q_{2i(n-1)+1}(x,y;\lambda)$ with the notation that P_k and Q_k are homogeneous polynomials of degree k. Thus, if $n = 2$, the full symmetric family coincides with (2).

The next theorem is the main theoretical result of [1]. The particular case with $p = q = 1$ and $n = 2$ was already proved in [5].

Theorem 10 ([1])**.** *There is a C^∞ Lyapunov function $W(x,y)$ for family* (2) *in a neighborhood of the origin whose formal Taylor expansion is given by the infinite jet*

$$\mathcal{W}(x,y) = \mathcal{J}^\infty W(x,y) = \frac{1}{2}y^2 + \sum_{\ell\geq 1} W_{2(q+s\ell)}(x,y) \tag{3}$$

where the W_k are (p,q)–quasihomogeneous polynomials of degree k and $W_{2(q+s)}(1,0) = \frac{1}{2(q+s)}$. More specifically, if $\hat{\mathcal{X}}_\lambda$ denotes the vector field associated to family (2) *then*

$$\hat{\mathcal{X}}_\lambda(\mathcal{W}(x,y)) = x^{2m}\sum_{j\geq 1}\eta_j(\lambda)x^{2js} \tag{4}$$

for some $m \in \mathbb{N}$ and polynomials $\eta_j \in \mathbb{R}[\lambda]$. Also the origin is a nilpotent center of system (2) *with parameters $\lambda = \lambda^*$ if and only if $\eta_j(\lambda^*) = 0$ for any $j \geq 1$.*

Remark 11. The formal series (3) is uniquely determined once the values of the constants $W_{2(q+s\ell)}(1,0)$ are fixed; we take $W_{2(q+s\ell)}(1,0) = 0$ for $\ell \geq 1$. From the proof of Theorem 10 given in [1] it follows, for the integer m appearing in (4), that $2m \geq 3(n-1)+2+(2\kappa-1)s$ where $\kappa = \min\{k \in \mathbb{Z} : 3(n-1)+2+(2k-1)s > 0\}$.

Definition 12. The polynomials $\eta_j \in \mathbb{R}[\lambda]$ for $j \geq 1$ are called *focus quantities* associated to the origin of family (2).

The next result shows that monodromic nilpotent family (2) enjoys the Poincaré-Lyapunov like property. This claim was conjectured in [1] but, in fact, it is a straightforward consequence of the results given in [1] and [17].

Theorem 13. *The origin is a nilpotent center of the monodromic analytic family* (2) *if and only if there is a local analytic first integral $H(x,y)$ which can be selected to have the expansion $H(x,y) = y^2 + \cdots$.*

Now we are ready to extend the theory developed in [13]. We shall consider a family (1) parameterized by $\lambda \in \mathbb{R}^M$. We assume that (1) is analytic in (x,y,λ) on an open neighborhood of $(0,0,\lambda^*) \in \mathbb{R}^2 \times \mathbb{R}^M$ and that the origin is always monodromic for the full family. Then, in a sufficiently short transversal Σ to the flow of (1) with one endpoint at the origin, one can define a Poincaré first return map $\Pi : \Sigma \to \Sigma$. It is known that Π is analytic, see [16] or the modern references [3,11]. Introducing a coordinate h on Σ, we define the *displacement map* as $d(h;\lambda) = \Pi(h;\lambda) - h$ which is analytic for $|h|$ and $\|\lambda - \lambda^*\|$ sufficiently small. Therefore we can expand in Taylor series $d(h;\lambda) = \sum_{i\geq 1} v_i(\lambda)h^i$, where the coefficients $v_i(\lambda)$ are called *Poincaré-Lyapunov quantities* and they are analytic on a neighborhood of λ^*. The computation of Poincaré-Lyapunov quantities is rather difficult since we must be able to compute primitives of functions involving generalized trigonometric functions as explained in Section 2.1.

In the case that interest us, family (2) will be polynomial and parameterized by its coefficients λ. Under these hypotheses the $v_i \in \mathbb{R}[\lambda]$. Since the ring $\mathbb{R}[\lambda]$ is noetherian, the *Bautin ideal* $\mathcal{B} =< v_i(\lambda) : i \in \mathbb{N} >$ of the family (1), is finitely generated by the Hilbert basis theorem.

Definition 14. The *minimal basis* of the Bautin ideal $\mathcal{B}$ with respect to an ordered basis $B = \{v_1, v_2, v_3, \dots\}$ is the basis $M_{\mathcal{B}}$ defined in the following way:
(a) Start with $M_{\mathcal{B}} = \{v_p\}$, where v_p is the first non-zero element of B;
(b) Adjoin v_j to $M_{\mathcal{B}}$ if and only if $v_j \notin \langle M_{\mathcal{B}} \rangle$, for all $j \geq p+1$.

Since all the ideals of interest to us lie in a polynomial ring over a field by the Hilbert Basis Theorem they are finitely generated, hence admit a unique minimal basis.

If (1) with $\lambda = \lambda^\dagger$ has a focus at the origin then $v_1(\lambda^\dagger) = \cdots = v_{r-1}(\lambda^\dagger) = 0$ but $v_r(\lambda^\dagger) \neq 0$ for some index $r \geq 1$. This implies that, for λ near λ^*, the displacement map can be written as

$$d(h;\lambda) = \sum_{i=1}^{r-1} v_i(\lambda)h^i + v_r(\lambda)[1 + \psi(h,\lambda)]h^r$$

with some analytic function ψ. From here we can deduce that $r-1$ is an upper bound for the cyclicity of the focus at the origin perturbing within family (1), see for example Proposition 6.1.2 in [19]. We want to remark here that in the work [2] this kind of degenerate Andronov-Hopf bifurcation of small amplitude limit cycles from a focus at the origin in family (2) is analyzed.

On the contrary, when (1) with $\lambda = \lambda^*$ has a center at the origin one has $v_i(\lambda^*) = 0$ for all $i \in \mathbb{N}$ and, in this case, the displacement map can be expressed as

$$d(h,\lambda) = \sum_{j=1}^{k} v_{i_j}(\lambda)[1 + \psi_j(h,\lambda)]h^{i_j},$$

where $\{v_{i_1}, \dots, v_{i_k}\}$ is the minimal basis of the Bautin ideal and ψ_j are certain analytic functions, see [19,20]. Now after a repeated application of a Rolle's Theorem kind of argument (see Lemma 6.1.6 and Theorem 6.1.7 of [19]) it is proved that the cyclicity of the center at the origin for $\lambda = \lambda^*$, with respect to perturbation within the family (1), is at most the cardinality of the minimal basis of the Bautin ideal minus one. Then we have the following theorem.

Theorem 15. *Let $\{v_{i_1}, \dots, v_{i_k}\} \subset \mathbb{R}[\lambda]$ be a minimal basis of the Bautin ideal $\mathcal{B}$ associated to the origin of family* (1) *with a monodromic nilpotent singularity at the origin. Then the cyclicity of any center at the origin in* (1) *is at most $k-1$.*

As mentioned before, the sequence of focus quantities $\{\eta_j(\lambda)\}_{j\in\mathbb{N}}$ of the monodromic nilpotent family (2) is easier to compute than its associated Poincaré-Lyapunov quantities sequence $\{v_j(\lambda)\}_{j\in\mathbb{N}}$. In [13] is described the relationship between these two sequences for the special monodromic family (1) where P and Q are nonlinear homogeneous polynomials of degree odd. Now we present this relationship for the larger family (2) with an analogous proof to that of Theorem 6.2.3 in [19] but with the technical differences associated to the degeneracy of the nilpotent monodromic singularity. This result is crucial to finally find how to bound the cyclicity of the centers inside the polynomial family (2) using focus quantities.

Theorem 16. *Let $\{v_j(\lambda)\}_{j\in\mathbb{N}}$ and $\{\eta_j(\lambda)\}_{j\in\mathbb{N}}$ be the sequence of polynomial Poincaré-Lyapunov quantities and focus quantities associated to the monodromic nilpotent singularity at the origin of a polynomial family* (2) *parameterized by λ. Consider the ideal $\mathcal{I}_k = \langle \eta_1, \ldots, \eta_k\rangle$ in $\mathbb{R}[\lambda]$. Let $m \in \mathbb{N}$ be the number defined in* (4)*. Then there exist positive real numbers w_k independent of the parameters λ such that:*

(i) $v_1 = v_2 = \cdots = v_{2(m+s)-3n+2} = 0$ *and* $v_{2(m+s)-3n+3} = w_1\eta_1$*;*

(ii) for $k \in \mathbb{N}$ with $k \geq 2$: $v_{2(m+(k-1)s)-3n+j} \in \mathcal{I}_{k-1}$ *for* $j = 3, \ldots, 2s+1$ *and* $v_{2(m+ks)-3n+2} - w_k\eta_k \in \mathcal{I}_{k-1}$.

A direct consequence of Theorem 16 is that the Bautin ideal $\mathcal{B}$ associated to the monodromic nilpotent singularity at the origin of a polynomial family (2) can be expressed in terms of both focus quantities and Poincaré-Lyapunov quantities. More specifically, $\mathcal{B} = \langle v_k : k \in \mathbb{N}\rangle = \langle \eta_k : k \in \mathbb{N}\rangle$ as ideals in $\mathbb{R}[\lambda]$. Additionally, if $\{v_{k_1}, \ldots, v_{k_\ell}\}$ and $\{\eta_{j_1}, \ldots, \eta_{j_r}\}$ are two minimal bases for $\mathcal{B}$, then they have the same cardinality so that $r = \ell$. All these relations together with Theorem 15 imply finally the following result.

Theorem 17. *Let $\{\eta_{k_1}, \ldots, \eta_{k_r}\}$ be the minimal basis of the Bautin ideal formed by focus quantities associated to the monodromic nilpotent singularity at the origin of a polynomial family* (2)*. Then the cyclicity of any center of* (2) *at the origin with respect to perturbation inside* (2) *is at most $r - 1$.*

After Theorem 17, it only remains to look for a method that allows us to find (at least in some cases) the elements of the minimal basis of the Bautin ideal $\mathcal{B}$. We describe it in the forthcoming Section 2.2 and summarize here in both cases when the Bautin ideal $\mathcal{B}$ is radical or when it is not.

From now we shall use the following notation. For a field $\mathbb{K}$ we denote by $\mathbf{V}(I) \subset \mathbb{K}^M$ the affine variety associated to a polynomial ideal $\mathcal{I} = \langle p_1(\mathbf{x}), \ldots, p_r(\mathbf{x})\rangle$ in $\mathbb{K}[\mathbf{x}]$ with $\mathbf{x} \in \mathbb{K}^M$. Thus $\mathbf{V}(I)$ is the set of common zeros in $\mathbb{K}^M$ of all elements of $\mathcal{I}$ when they are viewed as functions from $\mathbb{K}^M$ into $\mathbb{K}$.

Theorem 18. *Assume that $\{\eta_{k_1}, \ldots, \eta_{k_r}\}$ is a minimal basis of the ideal $\mathcal{I}_{j_r} \subseteq \mathcal{B}$ where $\mathcal{B}$ is the Bautin ideal associated to a polynomial family* (2)*. Suppose that $\mathcal{I}_{j_r}$ is radical and that the equality of varieties $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{j_r})$ holds in $\mathbb{C}^M$. Then $\mathcal{B} = \mathcal{I}_{j_r}$ and, in particular, the cyclicity of any center at the origin in* (2) *is at most $r - 1$.*

When the center problem has been already solved and we know that the center variety $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{j_r})$ but $\mathcal{I}_{j_r}$ is not radical one can obtain an upper bound on the cyclicity in some subset of the center variety as follows. It is based on some ideas from [9] and its proof is given [13]. We state it applied to family (2).

Theorem 19. *Assume that the nilpotent center problem at the origin of family* (2) *has been solved and its center variety* $\mathbf{V}(\mathcal{B})$ *satisfies that* $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{j_r})$ *as varieties in* $\mathbb{C}^M$. *Let* $\{\eta_{j_1}, \ldots, \eta_{j_r}\}$ *be a minimal basis of* $\mathcal{I}_{j_r}$ *and suppose a primary decomposition of* $\mathcal{I}_{j_r}$ *can be written as* $\mathcal{I}_{j_r} = \mathcal{R} \cap \mathcal{N}$ *where* $\mathcal{R}$ *is the intersection of the ideals in the decomposition that are prime and* $\mathcal{N}$ *is the intersection of the remaining ideals in the decomposition. Then for any system of family* (2) *corresponding to* $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$, *the cyclicity of the center at the origin is at most* $r-1$.

It is worth to emphasize here that it is possible for ideals I and J in $\mathbb{R}[\lambda]$ to have the equality of varieties $\mathbf{V}(I) = \mathbf{V}(J)$ in $\mathbb{R}^M$ but the inequality $\mathbf{V}(I) \neq \mathbf{V}(J)$ in $\mathbb{C}^M$. Thus it is important to know whether the equality $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{j_r})$ that holds in $\mathbb{R}^M$, also holds in $\mathbb{C}^M$. In the forthcoming Proposition 26 we show that when (2) is Hamiltonian or time-reversible under $(x, y, t) \mapsto (x, -y, -t)$ the former is true.

Remark 20. In order to check whether the upper bound on the cyclicity obtained via Theorem 18 or Theorem 19 for a center at the origin of family (2) with $\lambda = \lambda^* \in \mathbf{V}(\mathcal{B})$ is sharp we will specify a concrete perturbation of it within family (2) by means of an analytic curve $\varepsilon \mapsto \lambda(\varepsilon) \subset \mathbb{R}^M$ in the parameter space with $\lambda(0) = \lambda^*$. If the perturbation is such that we can choose the focus quantities $\{\eta_{j_1}(\lambda(\varepsilon)), \ldots, \eta_{j_r}(\lambda(\varepsilon))\}$ that form a minimal basis of the Bautin ideal $\mathcal{B}$ of the perturbed system in the form

$$|\eta_{j_1}(\lambda(\varepsilon))| \ll |\eta_{j_2}(\lambda(\varepsilon))| \ll \cdots \ll |\eta_{j_{r^*}}(\lambda(\varepsilon))| \ll 1,$$

with $r^* \leq r$ and $\eta_{j_i}(\lambda(\varepsilon))\eta_{j_{i+1}}(\lambda(\varepsilon)) < 0$ for $i = 1, \ldots, r^*-1$, then by using standard arguments of bifurcation theory we get that r^*-1 small amplitude limit cycles can be made to bifurcate from the origin of (2) with $\lambda = \lambda^*$. Clearly if r^* reach the maximum value $r^* = r$ then we have proved that the above cyclicity is $r-1$.

Some applications of the developed theory are now presented. Consider family (2) with $(p, q) = (1, 3)$ and $s = 1$ so that $n = 4$. If only the first two terms in (2) are considered we get $\mathcal{X} = \mathcal{X}_2 + \mathcal{X}_4$ where $\mathcal{X}_k$ are $(1,3)$-quasihomogeneous polynomial vector fields of weighted degree k and $\mathcal{X}_2 = y\partial_x$. The resulting family is

$$\dot{x} = y + a_1x^5 + a_2x^2y, \quad \dot{y} = -x^7 + b_1x^4y + b_2xy^2. \tag{5}$$

Simple computation gives $\varphi(x) = (5a_1 + b_1)x^4 + O(x^5)$ so that $\hat{\beta} \geq 4$ according with Proposition 6.

The nilpotent center problem for family (5) was solved in [1] obtaining that the origin is a center if and only if either $b_1 + 5a_1 = a_2 + b_2 = 0$ and the system is hamiltonian or $a_1 = b_1 = 0$ and the system is time-reversible under the symmetry $(x, y, t) \mapsto (x, -y, -t)$. Here we will solve the cyclicity problem for family (5).

Theorem 21. *The smallest cyclicity upper bound that applies to the entire center variety associated to any center at the origin in family* (5) *is 1. Moreover, there are specific centers belonging to* (5) *having cyclicity 1.*

In the second example we consider family (2) with $(p, q) = (1, 3)$ and $s = 3$, hence $n = 6$. Taking the first vector fields in (2) one has $\mathcal{X} = \mathcal{X}_2 + \mathcal{X}_8$ where $\mathcal{X}_k$ are $(1,3)$-quasihomogeneous polynomials of weighted degree k and $\mathcal{X}_2 = y\partial_x$. The associated differential system to $\mathcal{X}$ becomes the family

$$\dot{x} = y + a_1x^9 + a_2x^6y + a_3x^3y^2 + a_4y^3, \quad \dot{y} = -x^{11} + b_1x^8y + b_2x^5y^2 + b_3x^2y^3. \tag{6}$$

The parameters are $\lambda = (a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in \mathbb{R}^7$. Also it is easy to check that $\varphi(x) = (9a_1 + b_1)x^8 + O(x^9)$ so that $\hat{\beta} \geq 8$ in agreement with Proposition 6. The nilpotent center problem at the origin of (6) was solved in [1]. More specifically they prove that the origin is a center if and only if one of the following conditions hold:

(i) (Hamiltonian case) $9a_1 + b_1 = b_2 + 3a_2 = a_3 + b_3 = 0$;
(ii) $b_1 = -9a_1$, $a_3 = -a_1(b_2 - 6a_2 + 54a_1^2)$, $b_3 = 3a_1(b_2 + 18a_1^2)$.

Now we will solve the cyclicity problem for family (6).

Theorem 22. *The smallest cyclicity upper bound that applies to the entire center variety associated to any center at the origin in family* (6) *is 2. Moreover, there are specific centers belonging to* (6) *having cyclicity 2.*

In our last example we study family (2) with $(p, q) = (1, 1)$ and $n = 2$, hence $s = 1$. We take (2) with $\mathcal{X} = \sum_{i \geq 0} \mathcal{X}_{2i}$ where $\mathcal{X}_k$ are $(1, 1)$-quasihomogeneous polynomials of weighted degree k, $\mathcal{X}_0 = y\partial_x$, and the second component of $\mathcal{X}_2$ contains the monomial $-x^3$. In other notation, $\mathcal{X}_k$ are homogeneous polynomial vector fields of degree $k+1$. In order to obtain a family with a number of parameters not too high we shall study the following particular case: $\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_2 + \mathcal{X}_4$ with $\mathcal{X}_2 = Q_3(x, y; \lambda)\partial_5$ and $\mathcal{X}_4 = P_5(x, y; \lambda)\partial_x$ with Q_3 and P_5 homogeneous polynomials of degrees 3 and 5 respectively. The associated differential system to such a $\mathcal{X}$ is

$$\begin{aligned} \dot{x} &= y + Ax^5 + Bx^4y + Cx^3y^2 + Dx^2y^3 + Exy^4 + Fy^5, \\ \dot{y} &= -x^3 + Gx^2y + Hxy^2 + Iy^3. \end{aligned} \tag{7}$$

The parameters of the family are $\lambda = (A, B, C, D, E, F, G, H, I) \in \mathbb{R}^9$. Easy calculations yield $\varphi(x) = Gx^2 + O(x^3)$ so that $\hat{\beta} \geq 2$ agreeing with Proposition 6. First we will solve the center problem for this family.

Theorem 23. *The origin is a nilpotent center of family* (7) *if and only if* $A = C = E = G = I = 0$, *in which case the system is time-reversible with respect to the involution* $(x, y, t) \mapsto (x, -y, -t)$.

Lastly we analyze the nilpotent cyclicity problem for the origin of the quintic family (7).

Theorem 24. *For any system in the family* (7) *corresponding to a parameter value* $\lambda = (A, B, C, D, E, F, G, H, I) \in \mathbf{V}(\mathcal{B}) \setminus \{0\} \subset \mathbb{R}^9$, *the cyclicity of the nilpotent center at the origin is at most* 10. *Moreover there are systems in* (7) *with a center at the origin from which* 7 *small amplitude limit cycles bifurcate from the origin.*

After Theorem 24, the cyclicity upper bound problem for the origin of family (7) is completely solved if we find a bound on it when $\lambda = 0$, that is, we obtain an upper bound on the cyclicity of the nilpotent center at the origin in system $\dot{x} = y$, $\dot{y} = -x^3$ perturbing it within family (7).

From the noetherian property of polynomial ideals we know that the ascending chain of ideals $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_{\kappa-1} \subset \mathcal{I}_\kappa = \mathcal{B}$ stabilizes for some integer $\kappa \geq 1$. The computations made in the proof of Theorem 24 show that, for the origin in family (7), one has $\mathcal{I}_{12} = \mathcal{I}_j$ for $13 \leq j \leq 20$. So there are strong evidences for stating the following conjecture which is based on the conjecture $\kappa = 12$ or equivalently $\mathcal{B} = \mathcal{I}_{12}$.

Conjecture 25. *An upper bound for the cyclicity of the nilpotent center at the origin in system* $\dot{x} = y$, $\dot{y} = -x^3$ *perturbing it within family* (7) *is* 10.

2. Some preliminaries and background

2.1. **How to compute Poincaré-Lyapunov quantities.** The details of this subsection are explained, for instance, in [3, 11].

We use the function F of Theorem 1 and first we perform the analytic change of variables $(x, y) \mapsto (x, y - F(x))$ followed by the rescaling of the coordinates $(x, y) \mapsto (\xi\, x, -\xi\, y)$ with $\xi = (-1/a)^{1/(2-2n)}$ where n is the Andreev number of (2) and the constant $a \in \mathbb{R}$ is also defined in Theorem 1. The composition of these changes bring (2) into

$$\dot{x} \,=\, y\,(-1 + \tilde{P}(x, y; \lambda)), \quad \dot{y} \,=\, \hat{f}(x; \lambda) + y\,\hat{\varphi}(x; \lambda) + y^2\,\tilde{Q}(x, y; \lambda), \tag{8}$$

where $\tilde{P}(0, 0; \lambda) = 0$, $\hat{f}(x; \lambda) = x^{2n-1} + \cdots$ and either $\hat{\varphi}(x; \lambda) \equiv 0$ or $\hat{\varphi}(x; \lambda) = b\,x^{\hat{\beta}} + \cdots$ with $\hat{\beta} \geq n$ taking into account Proposition 6.

Now we use the so-called *generalized trigonometric functions* defined by Lyapunov [16] as the unique solution $x(\theta) = \operatorname{Cs}\theta$ and $y(\theta) = \operatorname{Sn}\theta$ of the Cauchy problem $\frac{dx}{d\theta} = -y$, $\frac{dy}{d\theta} = x^{2n-1}$ with initial condition $(x(0), y(0)) = (1, 0)$. We perform the generalized polar blow–up $(x, y) \mapsto (r, \theta)$ defined by

$$x \,=\, r\operatorname{Cs}\theta, \qquad y \,=\, r^n \operatorname{Sn}\theta \tag{9}$$

embedding a neighborhood of the origin in the (x, y)-plane into a cylinder $C = \{(r, \theta) \in \mathbb{R} \times \mathcal{S}^1\}$ with $|r|$ sufficiently small and where $\mathcal{S}^1 = \mathbb{R}/T\mathbb{Z}$ being T the minimal period of the generalized trigonometric functions. Also, since $\hat{\beta} \geq n$ or $\hat{\varphi}(x; \lambda) \equiv 0$, (9) transforms the analytic family (8) into

$$\dot{r} \,=\, \mathcal{R}(r, \theta; \lambda) = \tilde{p}(\theta)\, r^{n+1} + O(r^{n+2}), \quad \dot{\theta} \,=\, \Theta(r, \theta; \lambda) = r^{n-1} \,+ O(r^n). \tag{10}$$

In summary we have transformed family (2) into an ordinary differential equation defined on the cylinder C having the form

$$\frac{dr}{d\theta} \,=\, \mathcal{F}(r, \theta; \lambda), \tag{11}$$

where $\mathcal{F}(r, \theta; \lambda)$ is an analytic function on C and $\mathcal{F}(0, \theta) \equiv 0$ for all $\theta \in \mathcal{S}^1$. In this way, the singularity at the origin of (2) is transformed into the circle $\{r = 0\}$, a particular periodic (constant) solution of (11).

From (11) the Poincaré first return map Π can be defined by $\Pi(h; \lambda) = \Psi(T; h, \lambda)$, where $\Psi(\theta; h, \lambda)$ is the unique solution of the Cauchy problem formed by (11) with initial condition $\Psi(0; h, \lambda) = h$. Observe that Π is an analytic diffeomorphism in a neighborhood of $h = 0$ and that periodic orbits of (2) near $(x, y) = (0, 0)$ correspond to fixed points of Π, hence to zeros of the displacement map $d(h; \lambda) = \Pi(h; \lambda) - h = \sum_{i \geq 1} v_i(\lambda) h^i$. Writing $\Psi(\theta; h, \lambda) = \sum_{i \geq 1} \Psi_i(\theta, \lambda) h^i$, we obtain $v_1(\lambda) = \Psi_1(T, \lambda) - 1$ and $v_i(\lambda) = \Psi_i(T, \lambda)$ for $i \geq 2$.

2.2. **How to obtain a minimal basis of the Bautin ideal.** Now we describe a method that is particularly well designed for our polynomial family (2). This procedure is called *Approach I* in the paper [13]. See also [12] for a discussion in the nondegenerate center problem.

As starting point we assume the center problem for (2) has been solved, hence we know that the center variety is

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{k_r}), \tag{12}$$

and that $\{\eta_{k_1}, \dots, \eta_{k_r}\}$ is a minimal basis of $\mathcal{I}_{k_r}$. The main difficulty that arises now is that (12) is an equality of varieties in the real parameter space $\mathbb{R}^M$ and therefore we cannot extract any relation between the ideals $\mathcal{B}$ and $\mathcal{I}_{k_r}$ nor between their radicals $\sqrt{\mathcal{B}}$ and $\sqrt{\mathcal{I}_{k_r}}$ because the field $\mathbb{R}$ is not algebraically closed. To try to solve this problem (at least partially) we move it to the complex setting. This can be done because family (2) can be viewed as a vector field on $\mathbb{C}^2$ whose parameters λ are in $\mathbb{C}^M$. The step-by-step construction of the formal series $\mathcal{W}(x, y)$ giving in (3) that satisfying (4) does not depend on whether $(x, y) \in \mathbb{R}^2$ or $(x, y) \in \mathbb{C}^2$. Actually that construction produces the same polynomial sequence $\{\eta_j(\lambda)\}_{j\in\mathbb{N}}$ of focus quantities. Clearly $\mathcal{W}(x, y)$ is a formal first integral for system (2) with $\lambda = \lambda^* \in \mathbb{C}^M$ if and only if $\eta_k(\lambda^*) = 0$ for all $k \geq 1$.

Thus the key point we have to answer is to describe whether the equality (12) that holds in $\mathbb{R}^M$, also holds in the complex space $\mathbb{C}^M$. There is a wide class of complex systems for which the former is true as we show in the following proposition.

Proposition 26. *Consider the complex polynomial family* (2) *on* $\mathbb{C}^2$ *with complex parameters* $\lambda \in \mathbb{C}^M$. *Let* $\mathbf{V}(\mathcal{B}) \subset \mathbb{C}^M$ *be the complex center variety associated to the origin of* (2). *Then the equality of varieties* (12) *holds in* $\mathbb{C}^M$ *if, for any* $\lambda^* \in \mathbf{V}(\mathcal{B})$, *one of the following situations occur:*

- *(i) System* (2) *with* $\lambda = \lambda^*$ *is Hamiltonian;*
- *(ii) System* (2) *with* $\lambda = \lambda^*$ *is time-reversible under* $(x, y, t) \mapsto (x, -y, -t)$*;*
- *(iii) There is a formal first integral* $H \in \mathbb{C}[[x, y]]$ *of system* (2) *with* $\lambda = \lambda^*$.

We continue assuming that (12) holds in $\mathbb{C}^M$ and that we are lucky in the sense that additionally $\mathcal{I}_{k_r} = \sqrt{\mathcal{I}_{k_r}}$, that is, the ideal $\mathcal{I}_{k_r}$ is radical. The first assumption allows us to use the Strong Hilbert Nullstellensatz which means that, in the complex space, $\mathbf{V}(I) = \mathbf{V}(J)$ if and only if $\sqrt{I} = \sqrt{J}$. Under these assumptions we have $\mathcal{B} \subset \sqrt{\mathcal{B}} = \sqrt{\mathcal{I}_{k_r}} = \mathcal{I}_{k_r}$. Therefore, since by definition $\mathcal{I}_{k_r} \subset \mathcal{B}$, we conclude that $\mathcal{B} = \mathcal{I}_{k_r}$. Now we are ready to apply Theorem 17 since we have proved that $\{\eta_{k_1}, \dots, \eta_{k_r}\}$ is the minimal basis of $\mathcal{B}$ too.

Remark 27. Given a polynomial ideal $\mathcal{I}$, we can use the routine `minAssChar` in the `primdec.lib` library of SINGULAR to obtain the prime decomposition of $\sqrt{\mathcal{I}}$. Also, in order to check whether a polynomial ideal $\mathcal{I}$ is radical or not you can use the `primdecGTZ` or `primdecSY` routines in the `primdec.lib` library of SINGULAR for getting the primary decomposition of $\mathcal{I}$ from which we can see if that decomposition is actually prime. Another option is the `IsRadical` command in Maple.

3. Proofs of the results

In this section we give the proofs of all the main results of this work.

3.1. Proof of Proposition 6.

The family of vector fields (2) can be written as

$$\dot{x} = y + P(x, y) = y + \sum_{k \geq 2np-q} A_k(x, y), \quad \dot{y} = Q(x, y) = \sum_{k \geq (2n-1)p} B_k(x, y),$$

where A_k and B_k are (p,q)–quasihomogeneous polynomials of weighted degree k and $B_{(2n-1)p}(x,0) = -x^{2n-1}$.

Let $y = \hat{F}(x)$ with $\hat{F}(0) = 0$ be the unique solution of $y + A_k(x,y) = 0$. Clearly $F(x)$ exists and is analytic at the origin from the Implicit Function Theorem. Performing implicit derivation m times with respect to x one can see that the derivative $\hat{F}^{(m)}(0) = 0$ when $\frac{\partial^m A_k}{\partial x^m}(0,0) = 0$. On the other hand, $\frac{\partial^m A_k}{\partial x^m}(x,y)$ is a (p,q)–quasihomogeneous polynomials of weighted degree $d_m = k - mp$. From now we specialize on the case $k = 2np - q$ so that $d_m = 2np - q - mp = np - q + (n-m)p \geq 1 + (n-m)p$. Thus we deduce $d_n \geq 1$ and, consequently, $\hat{F}^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n$. We conclude that $\hat{F}(x) = O(x^r)$ with $r > n$.

Let $F(x)$ be the function defined in Theorem 1 for family (2). Then $y = F(x)$ with $F(0) = 0$ is the unique solution of $y + \sum_{k \geq 2np-q} A_k(x,y) = 0$ from which we get that $F(x) = O(x^r)$ with $r > n$ holds.

In the language of Theorem 1, we claim that $f(x) = Q(x, F(x)) = -x^{2n-1} + \cdots$ and therefore the Andreev number associated to the origin of (2) is just n. Let us prove the claim now. Since $F(x) = O(x^r)$ one has $B_k(x, F(x)) = \sum_{(i,j) \in S_k} b_{ij} x^i F^j(x)$ where the set S_k is defined by $S_k = \{(i,j) \in \mathbb{N}^2 : p\,i + qj = k\}$. Therefore $B_k(x, F(x)) = O(x^{\alpha_k})$ with $\alpha_k = \min\{i + rj : (i,j) \in S_k\}$.

Denoting $\#S_k$ the cardinality of the set S_k, we can rewrite B_k as

$$B_k(x,y) = \sum_{\ell=0}^{\#S_k - 1} b_\ell \, x^{i_{\max} - q\ell} y^{j_{\min} + p\ell} \tag{13}$$

where $(i_{\max}, j_{\min}) \in S_k$ and are defined as $i_{\max} = \max\{i \in \mathbb{N} : (i,j) \in S_k\}$ and $j_{\min} = \min\{j \in \mathbb{N} : (i,j) \in S_k\}$. Clearly $i_{\max} \leq k/p$ and if p divides k then $j_{\min} = 0$. Using the expression (13) we can write α_k as follows:

$$\alpha_k = \min\{i_{\max} + rj_{\min} + \ell(rp - q) : 0 \leq \ell \leq \#S_k - 1\} = i_{\max} + rj_{\min}$$

since $rp - q > np - q \geq 1$. The possible pairs $(i_{\max}, j_{\min})$ only can be

$$(i_{\max}, j_{\min}) \in \left\{ \left(\frac{k - \gamma q}{p}, \gamma \right) : 0 \leq \gamma \leq p - 1 \right\}$$

from which we see that

$$\alpha_k = \frac{k - \gamma q}{p} + r\gamma = \frac{k + \gamma(rp - q)}{p}$$

and, using again $rp - q \geq 1$, we check that $\alpha_{\bar{k}} > \alpha_k$ provided $\bar{k} > k$. Then the claim is proved because we conclude that $f(x) = Q(x, F(x)) = O(x^{\alpha_k})$ with $k = (2n-1)p$, or equivalently $f(x) = -x^{2n-1} + \cdots$ since $B_{(2n-1)p}(x,0) = -x^{2n-1}$. So the Andreev number of the origin in family (2) is n.

On the other hand,

$$\begin{aligned} \operatorname{div}\mathcal{X}(x,y) &= \frac{\partial}{\partial x}\left(\sum_{k \geq 2np-q} A_k(x,y) \right) + \frac{\partial}{\partial y}\left(\sum_{k \geq (2n-1)p} B_k(x,y) \right) \\ &= \sum_{k \geq (2n-1)p - q} C_k(x,y) \end{aligned}$$

where C_k are (p,q)–quasihomogeneous polynomials of weighted degree k. Hence $\varphi(x) = \mathrm{div}\mathcal{X}(x, F(x))$ is either identically zero or, using similar arguments to the previous one, $\varphi(x) = O(x^{\hat{\beta}})$ with $\hat{\beta} \geq n$. □

3.2. **Proof of Proposition 8.** First we observe that $q - p + 2is$ is even for any $i \geq 0$ since p and q are odd. Hence the vector field (2) is a sum of (p,q)-quasihomogeneous polynomial vector fields of even degrees. We claim that any (p,q)-quasihomogeneous polynomial vector field $\mathcal{X}_{2k}$ of degree $2k \geq 0$ is symmetric provided p and q are odd, hence we finish the proof.

Now we prove the claim. By definition $\mathcal{X}_{2k} = A(x,y)\partial_x + B(x,y)\partial_y$ where

$$A(x,y) = \sum_{pi+qj=2k+p} a_{ij} x^i y^j, \quad B(x,y) = \sum_{pi+qj=2k+q} b_{ij} x^i y^j,$$

for certain coefficients $a_{ij}, b_{ij} \in \mathbb{R}$. Both diophantine conditions $p\,i + qj = 2k + p$ and $p\,i + qj = 2k + q$ imply that the exponents $(i,j) \in \mathbb{N}^2$ in any monomial $x^i y^j$ of A and B are such that $p\,i + qj$ is odd. It follows that the indexes i and j must have different parity provided p and q are odd. Therefore $i + j$ is odd and, in consequence, $\mathcal{X}_{2k}$ is symmetric. □

3.3. **Proof of Theorem 13.** From Theorem 10 we see that the origin is a nilpotent center of system (2) if and only if there is a formal first integral $\mathcal{W}(x,y)$ given by (3). From the results obtained by Mattei and Moussu in [17] the existence of a formal first integral implies the existence of a local analytic first integral around any isolated singularity of an analytic planar vector field. Therefore the theorem follows. □

3.4. **Proof of Theorem 16.** First, without loss of generality, we may assume that family (2) has been written into the form (8).

We will compare the value of the displacement map $d(h;\lambda)$ with the change in $\mathcal{W}$ in one turn about the singularity at the origin, starting from the point $(x,y) = (h,0)$ with $h > 0$ sufficiently small until the orbit reach again the x-axis. We denote $\Delta\mathcal{W}(h;\lambda)$ this change in $\mathcal{W}$, and we will compute it by integrating the orbital derivative $\hat{\mathcal{X}}_\lambda(\mathcal{W})$ along the solution of (8) satisfying the initial condition $(x,y) = (h,0)$. We denote this solution by $(x(t;h,\lambda), y(t;h,\lambda))$.

Defining $\tau = \tau(h)$ as the time needed to give one turn about the origin, the variation in $\mathcal{W}$ is expressed as

$$\begin{aligned} \Delta\mathcal{W}(h;\lambda) &= \int_0^{\tau(h)} \tfrac{d}{dt}(\mathcal{W}(x(t;h,\lambda), y(t;h,\lambda)))\,dt \\ &= \int_0^{\tau(h)} \sum_{k\geq 1} \eta_k(\lambda)\, x^{K(k)}(t;h,\lambda)\,dt, \end{aligned} \tag{14}$$

where $K(k) = 2(m+ks)$, according to Theorem 10. Since family (2) satisfies $\hat{\beta} \geq n$ and the changes of variables transforming it into (8) keep invariant both $\hat{\beta}$ and n, from (10) we have

$$\frac{d\theta}{dt} = \Theta(r,\theta;\lambda) = r^{n-1}(1 + O(r)) = r^{n-1}\left(1 + \sum_{j\geq 1} u_j(\theta;\lambda) r^j\right).$$

Thus, in a sufficiently small punctured neighborhood of $(x, y) = (0, 0)$ where clearly $r > 0$, we can change the variable of integration in (14) from t to the generalized polar angle θ since $\dot{\theta} > 0$ there. Also, expanding the solution $\Psi(\theta; h, \lambda)$ of (11) satisfying $\Psi(0; h, \lambda) = h$ in a power series of h yields $\Psi(\theta; h, \lambda) = \sum_{k\geq 1} \Psi_k(\theta; \lambda)h^k$ with $\Psi_1(0; \lambda) = 1$ and $\Psi_k(0; \lambda) = 0$ for $k \geq 2$. In fact, inserting this power series into (11) and equating the coefficients of like powers of h gives that $\Psi_1'(\theta; \lambda) = 0$ from which we deduce that $\Psi_1(\theta; \lambda) \equiv 1$.

In summary we have that

$$
\begin{aligned}
dt &= \frac{d\theta}{\Theta(\Psi(\theta; h, \lambda), \theta; \lambda)} \\
&= \frac{d\theta}{\left(\sum_{k\geq 1} \Psi_k(\theta; \lambda)h^k\right)^{n-1} \left[1 + \sum_{j\geq 1} u_j(\theta; \lambda) \left(\sum_{k\geq 1} \Psi_k(\theta; \lambda)h^k\right)^j\right]} \\
&= \frac{d\theta}{h^{n-1} \left(1 + \sum_{k\geq 2} \Psi_k(\theta; \lambda)h^{k-1}\right)^{n-1} \left[1 + \sum_{j\geq 1} u_j(\theta; \lambda) \left(\sum_{k\geq 1} \Psi_k(\theta; \lambda)h^k\right)^j\right]} \\
&= h^{1-n} \Big[1 + \sum_{j\geq 1} \widetilde{u}_j(\theta; \lambda)h^j\Big] d\theta.
\end{aligned}
$$

Using (9) we know that $x(t(\theta); h, \lambda) = r(\theta; h, \lambda)\mathrm{Cs}\,\theta = \mathrm{Cs}\,\theta \sum_{i\geq 1} \Psi_i(\theta; \lambda)h^i$. Hence

$$
\begin{aligned}
\Delta\mathcal{W}(h; \lambda) &= h^{1-n} \sum_{k\geq 1} \left[\int_0^T \mathrm{Cs}^{K(k)}\theta \Big[\sum_{i\geq 1} \Psi_i(\theta; \lambda)h^i\Big]^{K(k)} \Big[1 + \sum_{j\geq 1} \widetilde{u}_j(\theta; \lambda)h^j\Big] d\theta\right] \eta_k(\lambda) \\
&= h^{1-n} \sum_{k\geq 1} \left[\int_0^T \mathrm{Cs}^{K(k)}\theta \Big[h^{K(k)} + \sum_{j\geq 1} \widehat{u}_j(\theta; \lambda)h^{K(k)+j}\Big] d\theta\right] \eta_k(\lambda) \\
&= h^{1-n} \sum_{k\geq 1} \Big[w_k h^{K(k)} + \eta_{k,1}(\lambda)h^{K(k)+1} + \eta_{k,2}(\lambda)h^{K(k)+2} + \cdots\Big] \eta_k(\lambda)
\end{aligned}
$$

where

$$
w_k = \int_0^T \mathrm{Cs}^{K(k)}(\theta)\, d\theta > 0
$$

since $K(k) = 2(m + ks)$ is even.

For each $h > 0$, we define $\zeta > 0$ by

$$
\zeta = u(h) = \mathcal{W}(h, 0) = \sum_{\ell\geq 1} W_{2(q+s\ell)}(h, 0) = \frac{1}{2np} h^{2n}
$$

where in the last step we have used first the condition $W_{2(q+s)}(1, 0) = \frac{1}{2(q+s)}$ stated in Theorem 10 so that $W_{2(q+s)}(h, 0) = \frac{1}{2(q+s)} h^{2(q+s)/p} = \frac{1}{2np} h^{2n}$ and later Remark 11 from which $W_{2(q+s\ell)}(1, 0) = 0$ for $\ell \geq 1$.

Clearly the restriction to $h > 0$ assures that $\zeta = u(h)$ has an inverse $h = g(\zeta) = 2np \sqrt[2n]{\zeta}$. Then we have

$$
\begin{aligned}
\Delta h &= g(\zeta + \Delta\mathcal{W}) - g(\zeta) = 2np \left(\sqrt[2n]{\zeta + \Delta\mathcal{W}} - \sqrt[2n]{\zeta}\right) \\
&= 2np \left(\sqrt[2n]{\frac{1}{2np} h^{2n} + \Delta\mathcal{W}} - \frac{1}{\sqrt[2n]{2np}} h\right). \qquad (15)
\end{aligned}
$$

We claim that this expression of Δh can be expanded in powers of h because $\Delta\mathcal{W} = w_1 h^{1-n+K(1)} + \cdots$ and the leading exponent $1-n+K(1)$ is larger than $2n$. To prove the claim, first we recall that, from the proof of Theorem 10 in [1], we know that the leading term in the power series $\hat{\mathcal{X}}_\lambda(\mathcal{W}(x,y))$ contains the monomial $x^{\tilde{m}}$ where $\tilde{m} = 3(n-1)+2+(2k-1)s+2s$ for some $k \geq 0$. Therefore, $\tilde{m} \geq 3n+s+1$. Now, in Theorem 10 it is defined $m \in \mathbb{N}$ in such a way that $2(m+s) = \tilde{m}$. So we have $1-n+K(1) = 1-n+2(m+s) \geq 1-n+3n+s+1 = 2n+s+2 > 2n$ proving the claim.

From the performed analysis it follows that we can write (15) as $\Delta h = O(h^2)$ because

$$\begin{aligned}\Delta h &= 2np\left(\sqrt[2n]{\frac{1}{2np}h^{2n}\left(1+\frac{2np\,\Delta\mathcal{W}}{h^{2n}}\right)} - \frac{1}{\sqrt[2n]{2np}}h\right) \\ &= \frac{2np}{\sqrt[2n]{2np}}h\left(\sqrt[2n]{1+\frac{2np\,\Delta\mathcal{W}}{h^{2n}}}-1\right) = (2np)^{1-1/(2n)}\left(\frac{p\,\Delta\mathcal{W}}{h^{2n-1}}+....\right).\end{aligned}$$

where in the last step we used $\sqrt[2n]{1+x} = 1+\frac{x}{2n}+O(x^2)$. Substituting here the expression for $\Delta\mathcal{W}$ from above and absorbing positive constants into the constants w_k yields

$$\Delta h = h^{2-3n}\sum_{k\geq 1}\left[w_k\eta_k(\lambda)h^{K(k)} + \eta_k(\lambda)[\widetilde{\eta}_{k,1}h^{K(k)+1} + \widetilde{\eta}_{k,2}h^{K(k)+2}+\cdots]\right].$$

More explicitly

$$\begin{aligned}\Delta h = {} & w_1\eta_1(\lambda)h^{K(1)-3n+2} + \eta_1(\lambda)[\widetilde{\eta}_{1,1}h^{K(1)-3n+3} + \widetilde{\eta}_{1,2}h^{K(1)-3n+4}+\cdots] \\ & + w_2\eta_2(\lambda)h^{K(2)-3n+2} + \eta_2(\lambda)[\widetilde{\eta}_{2,1}h^{K(2)-3n+3} + \widetilde{\eta}_{2,2}h^{K(2)-3n+4}+\cdots] \\ & + w_3\eta_3(\lambda)h^{K(3)-3n+2} + \eta_3(\lambda)[\widetilde{\eta}_{3,1}h^{K(3)-3n+3} + \widetilde{\eta}_{3,2}h^{K(3)-3n+4}+\cdots]+\cdots\end{aligned}$$

Comparing this expression to $\Delta h = d(h;\lambda) = \sum_{k\geq 1} v_k(\lambda)h^k$ we obtain

- (i) $v_1 = v_2 = \cdots = v_{K(1)-3n+1} = 0$ and $v_{K(1)-3n+2} = w_1\eta_1$
- (ii) for $k \in \mathbb{N}$ with $k \geq 2$: $v_{K(k-1)-3n+j} \in \mathcal{I}_{k-1}$ for $j = 3,\ldots,K(k)-K(k-1)+1$ and $v_{K(k)-3n+2} - w_k\eta_k \in \mathcal{I}_{k-1}$.

Thus, the conclusion of the theorem is reached using that $K(k) = 2(m+ks)$. □

3.5. **Proof of Proposition 26.** Since $\mathcal{I}_{k_r} \subset \mathcal{B}$ by definition, we always have that $\mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{I}_{k_r})$ holds in $\mathbb{C}^M$. Hence, in order to check (12) we only have to check that the reverse inclusion $\mathbf{V}(\mathcal{I}_{k_r}) \subset \mathbf{V}(\mathcal{B})$ holds in $\mathbb{C}^M$. To prove that we must check whether for any $\lambda^* \in \mathbb{C}^M$ satisfying $\eta_{k_1}(\lambda^*) = \cdots = \eta_{k_r}(\lambda^*) = 0$ this implies that $\eta_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$, or equivalently this implies the existence of a formal first integral for (2) with $\lambda = \lambda^*$ in the complex setting.

Therefore, statement (iii) is true. Clearly statement (i) is a particular (polynomial) case of (iii).

To prove statement (ii), first we note that the invariance of (2) by $(x,y,t) \mapsto (x,-y,-t)$ implies that (2) has the form

$$\dot{x} = y(1+A(x,y^2)), \qquad \dot{y} = B(x,y^2). \tag{16}$$

The polynomial map $y \mapsto z = y^2$ and the time rescaling $t \mapsto \tau$ with $d\tau = y\,dt$ transform (16) into

$$x' = 1 + A(x, z), \quad z' = 2B(x, z) \tag{17}$$

where $' = d/d\tau$. Since now the origin is a regular point, by the Flow box Theorem, there is a holomorphic first integral of the form $z + \cdots$ on a neighborhood of the origin which is pulled back into a holomorphic first integral of the form $y^2 + \cdots$ for (2) on a neighborhood of the origin. Thus we fall again in case (iii) finishing the proof. □

3.6. **Proof of Theorem 21.** Remark 11 gives $m \geq 1$ in the formal series (4). Taking $m = 1$ as reference we find, calculating by means of a computer algebra system such as Maple or Mathematica, that up to a positive multiplicative constant the first two non vanishing focus quantities are

$$\begin{aligned} \eta_5(\lambda) &= 5a_1 + b_1, \\ \eta_6(\lambda) &= -35a_1a_2 + 35a_1^2b_1 - 5a_2b_1 + 7a_1b_1^2 - 40a_1b_2 - 6b_1b_2. \end{aligned}$$

Let $\tilde{\eta}_k(\lambda)$ denote the reduction of $\eta_k(\lambda)$ modulo the ideal $\mathcal{I}_{k-1}$ generated by the previous $\eta_j(\lambda)$ for $j = 1, \ldots, k-1$. In other words $\tilde{\eta}_k(\lambda)$ is the remainder of $\eta_k(\lambda)$ upon division by a Gröbner basis of the ideal $\mathcal{I}_{k-1} = \langle \eta_1(\lambda), \ldots, \eta_{k-1}(\lambda) \rangle$. Computations yield

$$\begin{aligned} \tilde{\eta}_5(\lambda) &= 5a_1 + b_1, \\ \tilde{\eta}_6(\lambda) &= b_1(a_2 + b_2), \\ \tilde{\eta}_j(\lambda) &= 0, \quad \text{for } j = 7, 8, 9. \end{aligned}$$

Therefore, from the solution given in [1] of the center problem, it follows that the center variety for (5) is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\langle \tilde{\eta}_5(\lambda), \tilde{\eta}_6(\lambda) \rangle)$. This variety is clearly the union of the two irreducible components $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{J}_1) \cup \mathbf{V}(\mathcal{J}_2)$ where $\mathcal{J}_1 = \langle a_1, b_1 \rangle$ and $\mathcal{J}_2 = \langle b_1 + 5a_1, a_2 + b_2 \rangle$.

Following Approach I we view (5) as a system on $\mathbb{C}^2$ with complex parameters $\lambda = (a_1, a_2, b_1, b_2) \in \mathbb{C}^4$. We have that, for any $\lambda^* \in \mathbf{V}(\mathcal{B}) \subset \mathbb{C}^4$, system (5) is either Hamiltonian or invariant under $(x, y, t) \mapsto (x, -y, -t)$. Therefore, from Proposition 26, the equality of varieties $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_6)$ holds in $\mathbb{C}^4$ where $\mathcal{I}_6 = \langle \tilde{\eta}_5(\lambda), \tilde{\eta}_6(\lambda) \rangle$. Hence we conclude that $\sqrt{\mathcal{B}} = \sqrt{\mathcal{I}_6}$.

Using a symbolic manipulator we may verify that $\mathcal{I}_6$ is a radical ideal. In summary we have proved that $\mathcal{B} = \mathcal{I}_6$ and that $\{\tilde{\eta}_5(\lambda), \tilde{\eta}_6(\lambda)\}$ is the minimal basis of $\mathcal{B}$ with respect to the natural ordered infinite basis of focal values. Therefore, we conclude by Theorem 17 that any center at the origin in family (5) has cyclicity at most one.

Now we will prove that one limit cycles can be made to bifurcate from a center at the origin in family (5). Take the parameters

$$\lambda^* = (a_1, a_2, b_1, b_2) = (a_1, a_2, -5a_1, -a_2) \in \mathbf{V}(\mathcal{J}_2)$$

and perturb to $\lambda(\varepsilon) = \lambda^* + (0, \varepsilon, \varepsilon^2, 0) \in \mathbb{R}^4$ with $a_1 > 0$. Then we obtain $\tilde{\eta}_5(\lambda(\varepsilon)) = \varepsilon^2$ and $\tilde{\eta}_6(\lambda(\varepsilon)) = -5a_1\varepsilon$. Thus, for ε sufficiently small, this perturbation produce one small amplitude limit cycle from the center according to Remark 20. So the upper bound of one is sharp. □

3.7. **Proof of Theorem 22.** Remark 11 gives $m \geq 1$ in the formal series (4). Taking $m = 1$, the first non vanishing focus quantities, up to a positive multiplicative constant, are

$$\begin{aligned}
\eta_2(\lambda) &= 9a_1 + b_1, \\
\eta_3(\lambda) &= -135a_1a_2 + 9a_3 + 135a_1^2b_1 - 9a_2b_1 + 15a_1b_1^2 - 54a_1b_2 - 4b_1b_2 + 9b_3, \\
\eta_4(\lambda) &= 5670a_1a_2^2 + 5670a_1^2a_3 - 540a_2a_3 - 1215a_1a_4 - 11340a_1^2a_2b_1 + 270a_2^2b_1 + \\
&\quad 1863a_1a_3b_1 - 135a_4b_1 - 1008a_1a_2b_1^2 + 75a_3b_1^2 - 5670a_1^3b_2 + 3375a_1a_2b_2 - \\
&\quad 198a_3b_2 - 2898a_1^2b_1b_2 + 183a_2b_1b_2 - 168a_1b_1^2b_2 + 423a_1b_2^2 + 23b_1b_2^2 + \\
&\quad 6885a_1^2b_3 - 360a_2b_3 + 1773a_1b_1b_3 + 50b_1^2b_3 - 138b_2b_3.
\end{aligned}$$

Let $\tilde{\eta}_k(\lambda) = \eta_k(\lambda) \mod \mathcal{I}_{k-1}$. Using Mathematica again we find that

$$\begin{aligned}
\tilde{\eta}_2(\lambda) &= 9a_1 + b_1, \\
\tilde{\eta}_3(\lambda) &= 9a_3 + 6a_2b_1 + 2b_1b_2 + 9b_3, \\
\tilde{\eta}_4(\lambda) &= -2a_3b_1^2 - 9a_3b_2 + 18a_2b_3 - 2b_1^2b_3 - 3b_2b_3, \\
\tilde{\eta}_j(\lambda) &= 0, \quad \text{for} \quad j = 5, 6, 7, 8, 9.
\end{aligned}$$

Let $\mathcal{I}_4 = \langle \tilde{\eta}_2(\lambda), \tilde{\eta}_3(\lambda), \tilde{\eta}_4(\lambda) \rangle$. Using the routine `minAssChar` in the `primdec.LIB` library of Singular we find that the prime decomposition of $\sqrt{\mathcal{I}_4}$. In short we obtain that $\sqrt{\mathcal{I}_4} = \mathcal{J}_1 \cap \mathcal{J}_2$ where

$$\begin{aligned}
\mathcal{J}_1 &= \langle \tilde{\eta}_2(\lambda), a_3 + b_3, 3a_2 + b_2 \rangle \\
\mathcal{J}_2 &= \langle \tilde{\eta}_2(\lambda), \tilde{\eta}_3(\lambda), p_1(\lambda), p_2(\lambda), p_3(\lambda), p_4(\lambda) \rangle
\end{aligned}$$

where

$$\begin{aligned}
p_1(\lambda) &= 2b_1^3 + 9b_1b_2 + 27b_3, \\
p_2(\lambda) &= 2a_3b_1^2 + 2b_1^2b_3 + 9a_3b_2 - 18a_2b_3 + 3b_2b_3, \\
p_3(\lambda) &= 3a_3^2b_1 - 9a_2a_3b_2 - 3a_3b_2^2 + 18a_2^2b_3 + 6a_3b_1b_3 + 3a_2b_2b_3 - b_2^2b_3 + 3b_1b_3^2, \\
p_4(\lambda) &= 54a_2^2a_3b_2 + 36a_2a_3b_2^2 + 6a_3b_2^3 - 108a_2^3b_3 - 54a_2^2b_2b_3 + 2b_2^3b_3 + 27a_3^3 + \\
&\quad 81a_3^2b_3 + 81a_3b_3^2 + 27b_3^3.
\end{aligned}$$

The origin of system (6) with $\lambda = \lambda^*$ is a center if and only if either $\lambda^* \in \mathbf{V}(\mathcal{J}_1)$, the hamiltonian variety, or $\lambda^* \in \mathbf{V}(\mathcal{J}_2)$. You can easily verify that this statement agrees with [1]. Thus the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{J}_1) \cup \mathbf{V}(\mathcal{J}_2) = \mathbf{V}(\sqrt{\mathcal{I}_4})$.

We are lucky since $\mathcal{I}_4$ is a radical ideal. Hence we have that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_4)$ as varieties in $\mathbb{R}^7$.

Now we extend (6) to a polynomial family on $\mathbb{C}^2$ with complex parameters given by $\lambda = (a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in \mathbb{C}^7$ and we claim that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_4)$ also holds in $\mathbb{C}^7$. The above is true because, for any $\lambda^* \in \mathbf{V}(\mathcal{B}) \subset \mathbb{C}^7$, either $\lambda^* \in \mathbf{V}(\mathcal{J}_1)$ and system (6) with $\lambda = \lambda^*$ is Hamiltonian or $\lambda^* \in \mathbf{V}(\mathcal{J}_2)$ and, as we will show before, system (6) with $\lambda = \lambda^*$ possesses a complex formal (actually holomorphic) first integral. Hence the claim follows by statements (i) and (iii) of Proposition 26.

When $\lambda^* \in \mathbf{V}(\mathcal{J}_2)$ we can take (6) with $b_1 = -9a_1$, $a_3 = -a_1(b_2 - 6a_2 + 54a_1^2)$ and $b_3 = 3a_1(b_2 + 18a_1^2)$. Then, following [1], the polynomial map $(x, y) \mapsto (u, v) = (g(x, y), y^2)$ with $g(x, y) = x^6 + 6a_1x^3y - (b_2 + 18a_1^2)y^2$ and the time rescaling $t \mapsto \tau$ with $d\tau = y\frac{\partial g}{\partial x}(x, y)dt$ transform (6) into a (linear) differential system having a regular point at $(u,v)=(0,0)$. Thus, applying the Flowbox Theorem we deduce the existence of a holomorphic first integral $\phi(u,v)$ of the linear differential system on a neighborhood of the origin. Therefore $H(x,y)=\phi(g(x,y),y^2)$ is a holomorphic first integral of system (6) with $\lambda=\lambda^*\in\mathbf{V}(\mathcal{J}_2)$ on a neighborhood of the origin.

In summary we have that $\mathcal{B}=\mathcal{I}_4$ and that $\{\tilde\eta_2(\lambda),\tilde\eta_3(\lambda),\tilde\eta_4(\lambda)\}$ is a minimal basis of $\mathcal{B}$. From Theorem 17 we deduce that any center at the origin in family (6) has cyclicity at most two.

Now we will prove that the center cyclicity bound of two is sharp. Starting at a hamiltonian center with parameters

$$\lambda^*=(a_1,a_2,a_3,a_4,b_1,b_2,b_3)=(a_1,a_2,a_3,a_4,-9a_1,-3a_2,-a_3)\in\mathbf{V}(\mathcal{J}_1)$$

we perturb to $\lambda(\varepsilon)=\lambda^*+(0,0,0,0,\varepsilon^3,\varepsilon,2a_1\varepsilon-\varepsilon^2)\in\mathbb{R}^7$ and we obtain $\tilde\eta_2(\lambda(\varepsilon))=\varepsilon^3$, $\tilde\eta_3(\lambda(\varepsilon))=-9\varepsilon^2+O(\varepsilon^3)$ and $\tilde\eta_4(\lambda(\varepsilon))=-6(54a_1^3-9a_1a_2+a_3)\varepsilon+O(\varepsilon^2)$. In consequence, if we take our initial center with parameters satisfying the inequality $54a_1^3-9a_1a_2+a_3<0$, when ε is sufficiently small, the proposed perturbation gives two limit cycles bifurcating from the center according to Remark 20. □

3.8. **Proof of Theorem 23.** Taking $m=1$ as reference, we compute, up to a positive multiplicative constant, the first non vanishing focus quantities obtaining

$$\begin{aligned}
\eta_2(\lambda)&=G,\\
\eta_3(\lambda)&=15A-4GH+9I,\\
\eta_4(\lambda)&=30C-30BG+70AG^2-240AH+23GH^2+50G^2I-138HI,\\
\eta_5(\lambda)&=-5670AB+378E+5670A^2G-588DDG+420CG^2-1098CH+\\
&\qquad 1140BGH-1512AG^2H+3555AH^2-174GH^3-2268BI+15162AGI-\\
&\qquad 1396G^2HI+2115H^2I+4284GI^2.
\end{aligned}$$

We do not list more focus quantities because they have a huge number of monomials but we have computed them up to $\eta_{20}(\lambda)$. Now we make the reductions $\tilde\eta_k(\lambda)=$

$\eta_k(\lambda) \mod \mathcal{I}_{k-1}$ and we obtain

$$
\begin{aligned}
\tilde{\eta}_2(\lambda) &= G, \\
\tilde{\eta}_3(\lambda) &= 5A + 3I, \\
\tilde{\eta}_4(\lambda) &= 5C + HI, \\
\tilde{\eta}_5(\lambda) &= 15E + 45BI + 8H^2 I, \\
\tilde{\eta}_6(\lambda) &= -1005EH + 1035DDI + 64H^3 I - 864I^3, \\
\tilde{\eta}_7(\lambda) &= 15525BE - 146760EH^2 - 698625FI + 22528H^4 I - 353808HI^3, \\
\tilde{\eta}_8(\lambda) &= -580260EH^3 - 2130030FHI + 88016H^5 I - 1104759EI^2 - 1399356H^2 I^3, \\
\tilde{\eta}_9(\lambda) &= 9592423125EF + 399482285000FH^2 I + 868820400H^6 I - \\
&\quad 1011148939650EHI^2 - 21833986840H^3 I^3 - 108628849872I^5, \\
\tilde{\eta}_{10}(\lambda) &= -I(1173984523951882500FH^3 + 2214089098936000H^7 + \\
&\quad 113482899792337691 25FI^2 - 426888548704839600H^4 I^2 + \\
&\quad 5617111596814104048HI^4), \\
\tilde{\eta}_{11}(\lambda) &= HI(1777599005883003160 00H^7 + 1549545393369631413460500FI^2 - \\
&\quad 5365068311282151386340 0H^4 I^2 + 696271693221672317864883HI^4), \\
\tilde{\eta}_{12}(\lambda) &= I^5(7635248857281752854994580 13H^3 - 6436345390917915853583353710I^2), \\
\tilde{\eta}_j(\lambda) &= 0, \quad \text{for} \quad j = 13, \ldots 20.
\end{aligned}
$$

Computing we check that $\tilde{\eta}_{10}(\lambda) \notin \sqrt{\mathcal{I}_9}$ but $\tilde{\eta}_j(\lambda) \in \sqrt{\mathcal{I}_{10}}$ for $j = 11, \ldots, 20$.

Now we perform the prime decomposition of $\sqrt{\mathcal{I}_{10}}$ and we get the surprisingly simple output $\sqrt{\mathcal{I}_{10}} = \langle A, C, E, G, I \rangle$. Therefore we have that $\lambda^* \in \sqrt{\mathcal{I}_{10}}$ if and only if $A = C = E = G = I = 0$, in which case the system is time-reversible (hence has a center at the origin since it is monodromic). Thus we conclude that the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\sqrt{\mathcal{I}_{10}}) = \mathbf{V}(\langle A, C, E, G, I \rangle)$. $\square$

3.9. **Proof of Theorem 24.** From the proof of Theorem 23 we know that the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\sqrt{\mathcal{I}_{10}}) \subset \mathbb{R}^9$ where $\sqrt{\mathcal{I}_{10}} = \langle A, C, E, G, I \rangle$ and that $\mathcal{I}_{12} = \mathcal{I}_j$ for $13 \leq j \leq 20$. Unfortunately $\mathcal{I}_{12} \neq \sqrt{\mathcal{I}_{12}}$ so we cannot use Approach I to try to establish the equality between $\mathcal{B}$ and $\mathcal{I}_{12}$. It is now that Theorem 19 can be used.

First we note that the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{12}) \subset \mathbb{R}^9$. Now we extend (7) to a polynomial family of vector fields on $\mathbb{C}^2$ with complex parameters $\lambda = (A, B, C, D, E, F, G, H, I) \in \mathbb{C}^9$ and we claim that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{I}_{12})$ also holds in $\mathbb{C}^9$. The claim follows just combining Theorem 23 together with statement (ii) of Proposition 26.

Secondly we perform the primary decomposition of $\mathcal{I}_{12}$ and we obtain $\mathcal{I}_{12} = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R} = \sqrt{\mathcal{R}}$ and $\mathcal{N} \neq \sqrt{\mathcal{N}} = \langle A, B, C, D, E, F, G, H, I \rangle$. From here and taking into account that $\mathbf{V}(\mathcal{N}) = \mathbf{V}(\sqrt{\mathcal{N}})$ we get that $\mathbf{V}(\mathcal{N}) = \{0\}$, that is, the variety $\mathbf{V}(\mathcal{N})$ reduces to just a point: the origin of the parameter space. Finally, taking into account that $\{\tilde{\eta}_2, \ldots, \tilde{\eta}_{12}\}$ is a minimal basis of $\mathcal{I}_{12}$ having cardinality 11 and using Theorem 19 we get 10 as an upper bound for the center cyclicity at the origin inside family (7).

Now we try to bifurcate the maximum possible number of limit cycles by using the strategy explained in Remark 20. First we take an arbitrary point in the center variety

$$\lambda^* = (A, B, C, D, E, F, G, H, I) = (0, B, 0, D, 0, F, 0, H, 0) \in \mathbf{V}(\mathcal{B})$$

and perturb it to $\lambda(\varepsilon) = \lambda^* + \xi(\varepsilon) \in \mathbb{R}^9$ with $\xi(\varepsilon) = (\xi_1(\varepsilon), \ldots, \xi_9(\varepsilon))$ analytic at $\varepsilon = 0$ and $\xi(0) = 0$. Using the notation $\xi_i(\varepsilon) = \sum_{j\geq 1} \xi_{i,j}\, \varepsilon^j$ for $i \in \{1, \ldots, 9\}$ we get, up to a positive multiplicative constant, $\tilde{\eta}_{12}(\lambda(\varepsilon)) = H^3 \xi_{9,1}^5 \varepsilon^5 + O(\varepsilon^6)$ and $\tilde{\eta}_{11}(\lambda(\varepsilon)) = H^8 \xi_{9,1}\varepsilon + O(\varepsilon^2)$. Therefore, if we want that $|\tilde{\eta}_{11}(\lambda(\varepsilon))| \ll |\tilde{\eta}_{12}(\lambda(\varepsilon))|$ for $|\varepsilon| \ll 1$, the only option is $H\xi_{9,1} = 0$ but after some computations we arrive to a contradiction with the condition $|\tilde{\eta}_9(\lambda(\varepsilon))| \ll |\tilde{\eta}_{10}(\lambda(\varepsilon))| \ll |\tilde{\eta}_{11}(\lambda(\varepsilon))| \ll |\tilde{\eta}_{12}(\lambda(\varepsilon))|$. Therefore we rule out the possibility of using $\tilde{\eta}_{12}(\lambda(\varepsilon))$ and we cannot prove using this process if the cyclicity upper bound is sharp or not.

We only have been able to check that there are perturbations $\lambda(\varepsilon)$ such that $|\tilde{\eta}_2(\lambda(\varepsilon))| \ll |\tilde{\eta}_3(\lambda(\varepsilon))| \ll |\tilde{\eta}_4(\lambda(\varepsilon))| \ll |\tilde{\eta}_5(\lambda(\varepsilon))| \ll |\tilde{\eta}_6(\lambda(\varepsilon))| \ll |\tilde{\eta}_7(\lambda(\varepsilon))| \ll |\tilde{\eta}_8(\lambda(\varepsilon))| \ll |\tilde{\eta}_9(\lambda(\varepsilon))|$ with $\tilde{\eta}_j(\lambda(\varepsilon))\, \tilde{\eta}_{j+1}(\lambda(\varepsilon)) < 0$ for $j = 2, \ldots, 8$. Therefore, we can assure that 7 limit cycles bifurcate from the origin with such a kind of perturbations. A concrete example of these perturbations is

$$\begin{aligned}
\xi_1(\varepsilon) &= -\frac{3}{5}\varepsilon + \frac{3}{5}\varepsilon^6 + \varepsilon^7, \\
\xi_2(\varepsilon) &= \varepsilon - \frac{661(-8120081313257 + 1317265448043\sqrt{187105})}{206478299757578520}\varepsilon^2 + \\
&\quad \frac{134(61685 + 201\sqrt{187105}}{20522175}\varepsilon^3, \\
\xi_3(\varepsilon) &= \frac{1}{100}\varepsilon\left(-20 + \frac{201(61685 + 201\sqrt{187105})}{547258}\varepsilon - 20\varepsilon^2\right), \\
\xi_4(\varepsilon) &= 0, \\
\xi_5(\varepsilon) &= \frac{-12256385 + 26934\sqrt{187105}}{6840725}\varepsilon^2 + \\
&\quad \frac{-3291971421351305341 + 8456036595116751\sqrt{187105}}{1032391498787892600}\varepsilon^3, \\
\xi_6(\varepsilon) &= 2\frac{-19172720765 + 16700841\sqrt{187105}}{161899438575}\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\xi_7(\varepsilon) &= -\varepsilon^8, \\
\xi_8(\varepsilon) &= -\frac{201(61685 + 201\sqrt{187105})}{10945160}\varepsilon + \varepsilon^2, \\
\xi_9(\varepsilon) &= \varepsilon - \varepsilon^6,
\end{aligned}$$

which gives

$$\begin{aligned}
\tilde{\eta}_2(\lambda(\varepsilon)) &= -\varepsilon^8 + O(\varepsilon^9),\\
\tilde{\eta}_3(\lambda(\varepsilon)) &= 5\varepsilon^7 + O(\varepsilon^8)\\
\tilde{\eta}_4(\lambda(\varepsilon)) &= -\varepsilon^6 + O(\varepsilon^7),\\
\tilde{\eta}_5(\lambda(\varepsilon)) &= 8\varepsilon^5 + O(\varepsilon^6),\\
\tilde{\eta}_6(\lambda(\varepsilon)) &= \frac{-201(-45224810623321807265 + 153139605754913943\sqrt{187105})}{3277980400293950240}\varepsilon^4\\
&\quad +O(\varepsilon^5),\\
\tilde{\eta}_7(\lambda(\varepsilon)) &= 27\frac{-473942189301051052627 + 30832996689294978720\sqrt{187105}}{590183806807078603}\varepsilon^3\\
&\quad +O(\varepsilon^4),\\
\tilde{\eta}_8(\lambda(\varepsilon)) &= 57132\frac{-3270270 + 6533\sqrt{187105}}{273629}\varepsilon^2 + O(\varepsilon^3),\\
\tilde{\eta}_9(\lambda(\varepsilon)) &= 556082500\frac{-35416659351 + 81896683\sqrt{187105}}{40353607}\varepsilon + O(\varepsilon^2).\square
\end{aligned}$$

References

[1] A. Algaba, C. García, and M. Reyes, *The center problem for a family of systems of differential equations having a nilpotent singular point*, J. Math. Anal. Appl. **340** (2008), 32–43.

[2] A. Algaba, C. García, and M. Reyes, *Local bifurcation of limit cycles and integrability of a class of nilpotent systems of differential equations*, Appl. Math. Comput. **215** (2009), 314323.

[3] M.J. Álvarez and A. Gasull, *Monodromy and stability for nilpotent critical points*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **15** (2005), 1253–1265.

[4] M.J. Álvarez and A. Gasull, *Generating limit cycles from a nilpotent critical point via normal forms*, J. Math. Anal. Appl. **318** (2006), 271-287.

[5] V.V. Amel'kin, N.A. Lukashevich and A.P. Sadovski, Nonlinear Oscillations in Second Order Systems, 1982. Minsk, BGY Lenin, B. I. Press (in Russian).

[6] A. Andreev, *Investigation on the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translations Amer. Math. Soc. **8** (1958) 187–207.

[7] V.I. Arnold and Y.S. Ilyashenko, *Ordinary differential equations*, Dynamical Systems I, Encyclopaedia Math. Sci., vol. 1, Springer, Berlin, 1988.

[8] J. Écalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Actualités Mathématiques, Hermann, Paris, 1992.

[9] B. Ferčec, V. Levandovskyy, V. G. Romanovski, and D. S. Shafer, *Bifurcation of Critical Periods of Polynomial Systems*, J. Differential Equations **259** (2015), 3825-3853.

[10] I.A. García, H. Giacomini, J. Giné and J. Llibre, *Analytic nilpotent centers as limits of nondegenerate centers revisited*, to appear.

[11] I.A. García, H. Giacomini and M. Grau, *Generalized Hopf bifurcation for planar vector fields via the inverse integrating factor*, J. Dyn. Differ. Equ. **23** (2011), 251–281.

[12] I.A. García, J. Llibre and S. Maza, *The Hopf cyclicity of the centers of a class of quintic polynomial vector fields*, J. Differential Equations **258** (2015), 1990–2009.

[13] I.A. García and D.S. Shafer, *Cyclicity of a class of polynomial nilpotent center singularities*, Discrete Contin. Dyn. Syst. **36** (2016), 2497–2520.

[14] H. Giacomini, J. Giné and J. Llibre, *The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems*, J. Differential Equations **227** (2006) 406–426.

[15] Y. Ilyashenko, *Finiteness Theorems for Limit Cycles*, Transl. Math. Monogr., vol. 94, Amer. Math. Soc., Providence, RI, 1991.

[16] A. M. Lyapunov, *Stability of motion.* Mathematics in Science and Engineering, Vol **30**. Academic Press, New York-London (1966).

[17] J.F. Mattei and R. Moussu, *Holonomie et intégrales premières*, Annales Scientifiques de l'École Normale Supérieure **13** (1980) 469–523.

[18] R. Moussu, *Symétrie et forme normale des centres et foyers dégénérés*, Ergodic Theory & Dynam. Systems **2** (1982), 241–251.

[19] V. G. Romanovski and D.S. Shafer, *The center and cyclicity problems: a computational algebra approach.* Birkhäuser Boston, Inc., Boston, MA, 2009.

[20] R. Roussarie, *Bifurcation of planar vector fields and Hilberts sixteenth problem*, Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998.

[1] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain

E-mail address: garcia@matematica.udl.cat